ISRAEL JOURNAL OF MATHEMATICS 96 (1996), 115–140

LOCALIZATION, COMPLETION AND THE AR PROPERTY IN NOETHERIAN P.I. RINGS

BY

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Dedicated to the memory of Shimshon Amitsur

ABSTRACT

Given a prime ideal P in a noetherian ring R we examine the following two properties: (1) P is Ore localizable. (2) The completion of R at P is Noetherian. For rings satisfying the 2nd layer condition a strong connection is discovered between (1) and (2) and consequently questions by Goldie and McConnell are answered. As a corollary we also obtain a new characterization for non-maximal primitive ideal P in R to satisfy (1), where R is the enveloping algebra of complex solvable finite dimensional Lie algebra

1. Introduction

Localization and completion at a prime ideal of a commutative ring are very basic and important techniques. Goldie [G] initiated a study of the completion at a prime ideal P of a non-commutative Noetherian ring R, primarily in order to compensate for the lack of Ore localizability at P. A natural question which he raised was: when is $\varprojlim_i R/P^i$ Noetherian? Furthermore, the question of relating Noetherian completion to the Ore localizability at the prime ideal P was left open. In [D, p. 196] McConnell made this question more explicit by asking whether the localizability at a prime ideal P is related to the Noetherian property of $\varprojlim_i R/P^i$. Related examples were given by McConnell [Mc1], Jordan [J1], and Small–Stafford [S^2].

Received February 22, 1995 and in revised form January 1, 1996

The purpose of the present paper is to explore the possible connection between these two notions. Surprisingly they are strongly related. One of our main positive results is the following Theorem.

THEOREM 2.4: *Let R be a Noetherian ring with the 2nd layer condition and P a prime ideal in R satisfying*

(1) *R/P is semi-primitive,*
(2) *$P \subseteq N$, where N is a localizable semi-prime ideal,*
(3) *$\varprojlim_i R/P^i$ is Noetherian.*

Then P is (Ore) localizable.

Remarks 1: For the notion of "2nd layer condition" (s.l.c. in short) we refer the reader to [GW] or [Ja]. We remark that Noetherian P.I. rings and enveloping algebras of solvable finite dimensional Lie algebras over $\mathbb{C}$, both satisfy s.l.c.

2. A similar theorem for a semi-prime "homogeneous" ideal P is given in the paper.

3. Examples in Section 3 show that one cannot drop any of the conditions in Theorem 2.4.

4. Condition (2) can be avoided in case R is a P.I. ring and $R = T(R)$, the trace ring of R (Theorem 3.7).

Our next result is a consequence of Theorem 2.4.

THEOREM 2.6: *Let g be a finite dimensional complex solvable Lie algebra and $R \equiv Ug$ its enveloping algebra. Let P be a non-maximal primitive ideal in R. Then the following conditions are equivalent:*

(1) *P is localizable,*
(2) *$\varprojlim_i R/P^i \equiv \hat{R}$ is Noetherian and P is strictly contained in a localizable prime ideal of R.*

Theorem 2.5 is a similar theorem to Theorem 2.4, with essentially the same proof, which deals with the (weak, topological) AR property. Using Theorem 2.5 one is able to characterize the AR property of an ideal, in a Noetherian P.I. ring, in terms of completion as follows:

THEOREM 4.6: *Let R be a local Noetherian P.I. ring and I an ideal in R. Then the following are equivalent:*

(1) *I satisfies the AR property,*

(2) $\varprojlim_i R\langle t\rangle/I^i\langle t\rangle$ *is Noetherian.*

Remarks: 1. We recall that by [RSS], $S = \{f \in R[t] \mid f \text{ is monic}\}$ is an Ore set in $R[t]$ and $R\langle t\rangle \equiv R[t]_s$ is a Jacobson ring.

2. A more elaborate version for R semi-local can be obtained.

3. This result should be compared with a result of Goldie [G] where a stronger assumption: $\text{gr}(I)$ being Noetherian, implies that I satisfies the strong AR property.

We now discuss possible converses to Theorem 2.4: Suppose that P is localizable, is the P-adic completion $\hat{R}$ Noetherian? Firstly we recall the following result:

THEOREM ([B1, Theorem 14])): *Let R be a left Noetherian P.I. ring and P a maximal ideal in R. Then* $\varprojlim_i R/P^i$ *is left Noetherian.*

The next step will naturally be K.dim $R/P = 1$, as shown in the next theorem.

THEOREM 5.1: *Let R be a left Noetherian P.I. ring and P a prime ideal in R. Suppose that*

(1) *P is a finitely generated right R-module,*
(2) K.dim $R/P = 1$,
(3) *P is left localizable.*

Then $\hat{R} \equiv \varprojlim_i R/P^i$ *is left Noetherian.*

Combined with Theorem 2.4 this leads to the following characterization of a localizable prime P with K.dim $R/P = 1$.

THEOREM 5.7: *Let R be a Noetherian P.I. ring and P a prime ideal in R with* K.dim $R/P = 1$. *Then the following are equivalent.*

(1) *P is localizable.*
(2) (i) *$P[t] \subseteq N, N$ is a localizable semi-prime ideal in $R[t]$, and*
    (ii) $\varprojlim_i R[t]/P^i[t]$ *is Noetherian.*

Remarks: 1. $P[t] \subseteq N$ can be replaced by $P[t] \subset N$.

2. If $R = T(R)$, then (i) and K.dim $R/P = 1$ are superfluous.

3. Instead of using $R[t]$ we could of course have assumed that R/P is semi-primitive.

The next step is therefore K.dim $R/P = 2$. Here, unfortunately, we face the following counterexample:

Example 6.6: There exists a prime Noetherian affine P. I. ring which is a finite module over its center and a localizable prime ideal P in R satisfying:

(1) p.i.deg$(R/P) =$ p.i.deg(R).

(2) K.dim $R/P = 2$.

(3) $\varprojlim_i R/P^i$ is not Noetherian (neither left nor right).

(4) $P \subset N$ where N is a maximal and localizable ideal.

(5) P does not satisfy the AR property in R and P_N does not satisfy the AR property in R_N.

(6) R_N is a prime local Noetherian P.I. ring which is a finite module over its center and $\varprojlim_i R_N/P_N^i$ is not Noetherian.

Remark: Example 6.6 provides a counterexample to a question of Jordan [J2, Remark 2, p. 233].

When summing up the answers to the questions raised at the beginning, we see that in case K.dim $R/P = 0$, the Noetherian property of $\hat{R}$ is weaker than the localizability of P. When K.dim $R/P = 1$, then the localizability of P is equivalent (under mild assumptions) to $\hat{R}$ being Noetherian. Now if K.dim $R/P = 2$, the localizability of P is a much weaker property than the Noetherian property of $\hat{R}$.

The paper is organized as follows. In Section 1 we have, in addition to the introduction, some definitions and preliminaries. In Section 2 we prove Theorems 2.4, 2.5 and 2.6. Section 3 provides examples showing that the assumptions of Theorem 2.4 cannot be weakened. We also prove here (Theorem 3.7), in the case where $T(R) = R$, an improved version of Theorem 2.4. We also prove some general results concerning localization (Propositions 3.5 and 3.6). In Section 4 we prove Theorem 4.6. Section 5 is devoted to the proofs of Theorems 5.1 and 5.7. In Section 6 we construct Example 6.6.

We now recall some definitions and notations. For standard unexplained terminology we refer to [Mc-R], [GW] and [Ro]. We recall that the prime ideals P, Q are said to be linked, and this relation is denoted by $P \rightsquigarrow Q$, if there exists an ideal A with $PQ \subseteq A \subset P \cap Q$, and such that $P \cap Q/A$ is a left torsionfree R/P-module as well as a right torsionfree R/Q-module (e.g. [GW]). By the term Noetherian we always mean (unless specified otherwise) left and right Noetherian. By K.dim R we always denote the *classical* Krull dimension of R. For a Noetherian P.I. ring this is equal to the usual Krull dimension. We shall denote by $\hat{R} \equiv \lim_{\leftarrow i} R/I^i$ and remark that this is defined in the cases where $\bigcap_i I^i \neq \{0\}$ by looking at $\bar{R} = R/\bigcap_i I^i$, $\bar{I} \equiv I/\bigcap_i I^i$ and taking $\hat{R} \equiv \hat{\bar{R}}$. Finally, $A \subset B$ means that A is properly contained in B.

2. Noetherian completion implies localizability

The main purpose of this section is to prove Theorems 2.4, 2.5 and 2.6. We begin with a lemma which will enable us to reduce the problem to the case $\bigcap_n P^n = \{0\}$.

LEMMA 2.1: *Let R be a Noetherian ring satisfying the s.l.c., and P a prime ideal in R. Suppose that*

(1) *$P \subseteq N$, N a localizable semi-prime ideal in R,*

(2) *$P/\bigcap_n P^n$ is left localizable in $R/\bigcap_n P^n$.*

Then P is left localizable in R.

Proof: Let $I = \{x \in R |\ txs = 0 \text{ for some } t, s \in \mathcal{C}(N)\}$. It is an easy exercise, using the right and left Ore condition of $\mathcal{C}(N)$, to show that I is a two-sided ideal in R. Let V be a prime ideal in R with $P \rightsquigarrow V$. By definition there exists an ideal A in R, $PV \subseteq A \subset P \cap V$ and $P \cap V/A$ is $R/P - R/V$ torsionfree bimodule. It is easy to see, using [GW, 12.17], that $\mathcal{C}(N) \subseteq \mathcal{C}(V)$. Consequently $I \subseteq P \cap V$ and therefore $I \subseteq A$. Next, since $\mathcal{C}(N)$ consists of regular elements in R/I we have $R/I \subseteq (R/I)_{\bar{N}}$, the latter being the localization of R/I with respect to $\mathcal{C}(N/I)$. Consequently by Jategaonkar's Theorem [GW, Theorem 12.8] we have that $\bigcap_i \bar{N}^i_{\bar{N}} = \{\bar{0}\}$. Hence $\bar{P} \equiv P/I$ satisfies $\bigcap_i \bar{P}^i = \{0\}$ and therefore $\bigcap_i P^i \subseteq I$. Consequently, using $I \subseteq A$, the link $P \cap V/A$ is preserved in $R/\bigcap_n P^n$. However, by the left localizability of $P/\bigcap_n P^n$ and [GW, Theorem 12.21], we have that $P/\bigcap_n P^n = V/\bigcap_n P^n$ and therefore $P = V$. Again, this shows by [GW, Theorem 12.21] that P is left localizable. ∎

Remark: As seen before, the implication: "$P \rightsquigarrow V \Rightarrow P = V$", shows, by [GW, Theorem 12.21], that P is left localizable; and this is frequently used in this paper.

PROPOSITION 2.2: *Let R be a Noetherian ring satisfying s.l.c. and P a prime ideal in R. Let N be a localizable semiprime ideal in R satisfying $\mathcal{C}(N) \subseteq \mathcal{C}(P)$ and denote by $X(P)$ the clique of P. Then*

(1) *$\bigcap_i (N^i + Q) = Q$, for each $Q \in X(P)$.*

(2) *Let* $P_1, P_2 \in X(P)$ *and* A *an ideal in* R *satisfying* $P_1P_2 \subset A \subseteq P_1 \cap P_2$, *with* $P_1 \cap P_2/A$ *is* $R/P_1 - R/P_2$ *torsionfree. Then* $\bigcap_i (N^i + A) = A$.

Proof: By [GW, Lemma 12.17] we have that $\mathcal{C}(N) \subset \mathcal{C}(Q)$ holds for every $Q \in X(P)$. Let $P_1, P_2 \in X(P)$ with $P_1 \rightsquigarrow P_2$. Hence there exists an ideal A in R satisfying $P_1P_2 \subseteq A \subset P_1 \cap P_2$ and $P_1 \cap P_2/A$ is a $R/P_1 - R/P_2$ torsion-free bimodule. Hence $\bar{R} \equiv R/A$ has an Artinian quotient ring with $\mathcal{C}_{\bar{R}}(0) = \mathcal{C}(\bar{P}_1) \cap \mathcal{C}(\bar{P}_2)$. Consequently, since $\mathcal{C}(\bar{N}) \subseteq \mathcal{C}(\bar{P}_1) \cap \mathcal{C}(\bar{P}_2)$, we have that $\bar{R} \subseteq \bar{R}_{\bar{N}}$, the latter being a semilocal ring satisfying s.l.c. So by Jategaonkar's Theorem [GW, Theorem 12.8] we have that $\bigcap_i \bar{N}^i_{\bar{N}} = \{\bar{0}\}$. Consequently, $\bigcap_i \bar{N}^i = \{\bar{0}\}$, that is $\bigcap_i (N^i + A) = A$. This proves (2). Also $\bar{N}_{\bar{N}} = \mathrm{Jac}(\bar{R}_{\bar{N}})$ so $\bar{N}_{\bar{N}}$ projects into the Jacobson radical of every homomorphic image of $\bar{R}_{\bar{N}}$. Consequently every ideal in $\bar{R}_{\bar{N}}$ is closed with respect to the topology induced by $\{\bar{N}^i_{\bar{N}}\}$. Let $Q \in X(P)$. In particular $\bar{Q}_{\bar{N}} = \bigcap_i (\bar{Q}_{\bar{N}} + \bar{N}^i_{\bar{N}})$. Hence

$$\bar{Q} = \bar{Q}_{\bar{N}} \cap \bar{R} = [\bigcap_i (\bar{Q}_{\bar{N}} + \bar{N}^i_{\bar{N}})] \cap \bar{R} \supseteq \bigcap_i (\bar{Q} + \bar{N}^i) \supseteq \bar{Q},$$

which implies that $\bar{Q} = \bigcap_i (\bar{Q} + \bar{N}^i)$, that is $Q = \bigcap_i (Q + N^i)$. ∎

We collect here, for the sake of completeness, several standard results about completions.

LEMMA 2.3: *Let* I, J *and* P *be ideals of the ring* R. *Suppose* P *is right (or left) finitely generated. Let* $\hat{R} \equiv \varprojlim R/P^i$. *Then*

(1) $\hat{I}\hat{J} \subseteq \widehat{IJ}$, *where* $\hat{I}$ *(respectively* $\hat{J}$*) is the closure of* I *(respectively* J*) in* $\hat{R}$.
(2) $(\hat{P})^n = \widehat{P^n} = \hat{R}P^n = P^n\hat{R}$ *for each* n.
(3) $\hat{P} \subseteq \mathrm{Jac}(\hat{R})$.

Proof: First we prove (1). Let $x \in \hat{I}$, $y \in \hat{J}$. Then $x = \lim x_n$, $x_n \in I$, $y = \lim y_n$, $y_n \in J$, where $x - x_n \in \widehat{P^n}$. We have

$$xy - x_ny_n = x(y - y_n) + (x - x_n)y_n \in \hat{R}\widehat{P^n} + \widehat{P^n}\hat{R} \subseteq \widehat{P^n}, \quad \text{for each } n.$$

Hence, since $x_ny_n \in IJ$ for each n, we get $xy \in \widehat{IJ}$.

Next we observe that by [Ro, Vol. I, p. 118] $\widehat{P^n} = \hat{R}P^n$ for each n. Consequently $\widehat{P^n} \subseteq (\hat{P})^n$. The other inclusion follows from (1).

Finally (3) is proved in [Ro, Vol. I, p. 117].

We are able now to prove the following:

THEOREM 2.4: *Let R be a Noetherian ring satisfying s.l.c. and P a prime ideal. Suppose that*

(1) *R/P is semiprimitive,*
(2) *$P \subseteq N$, when N is a localizable semi-prime ideal in R,*
(3) *$\lim_{\leftarrow i} R/P^i$ is Noetherian.*

Then P is localizable.

Proof: By Lemma 2.1, we may assume that $\bigcap_n P^n = \{0\}$. Hence $R \subset \hat{R} \equiv \lim_{\leftarrow i} R/P^n$. Let Q be a prime ideal satisfying $P \rightsquigarrow Q$. Let $PQ \subset A \subset P \cap Q$ with $P \cap Q/A$ a $R/P - R/Q$ torsionfree bimodule. By Proposition 2.2 we have $\bigcap_i (A + N^i) = A$ and $\bigcap_i (N^i + Q) = Q$. Consequently $\hat{A} \equiv \bigcap_i (A + \hat{P}^i)$ satisfies $\hat{A} \cap R = A$ and likewise $\hat{Q} \cap R = Q$. Clearly $\hat{P} \cap R = P$. Let $B \equiv \hat{P} \cap \hat{Q}/\hat{A}$. Then $B \cap (R/A) = P \cap Q/A$. Now by Lemma 2.3, $\hat{P}\hat{Q} \subseteq \widehat{PQ} \subseteq \hat{A}$, which implies that $\ell\text{-ann}_{\hat{R}} B \supseteq \hat{P}$ and $r\text{-ann}_{\hat{R}} B \supseteq \hat{Q}$. Also $B \cap (R/A) = P \cap Q/A$ implies that $(\ell\text{-ann}_{\hat{R}} B) \cap R \subseteq \ell\text{-ann}_R P \cap Q/A = P$.

Since $R/P \cong \hat{R}/\hat{P}$, we must have, using $\ell\text{-ann}_{\hat{R}} B \supseteq \hat{P}$, that $\ell\text{-ann}_{\hat{R}} B = \hat{P}$. Similarly $(r\text{-ann}_{\hat{R}} B) \cap R \subseteq Q$. However, $\hat{Q} \cap R = Q$ implies that $(r\text{-ann}_{\hat{R}} B) \cap R = Q$. Observe that $\hat{R}/\hat{P} \cong R/P$ is a Noetherian ring satisfying the s.l.c. So by applying [GW, Lemma 12.3] to the $\hat{R}/\hat{P} - \hat{R}/\hat{Q}$ bimodule B, there exist sub-bimodules B' and B'', with $B' \supset B''$ such that B'/B'' is a torsionfree left $\hat{R}/\hat{P}$-module as well as a torsionfree right $\hat{R}/M$-module for some prime ideal M in $\hat{R}$, $M \supseteq \hat{Q}$. By (1) and [GW, Theorem 7.16] we get that $\hat{R}/M$ is semi-primitive. Consequently $M \supseteq \text{Jac}(\hat{R}) \supseteq \hat{P}$ (the last inclusion is by Lemma 2.3). Now since $\hat{R}/\hat{P}$ satisfies the s.l.c., we may apply [GW, Theorem 12.4] to B'/B'', as a $\hat{R}/\hat{P} - \hat{R}/M$ bimodule, and get that K.dim $\hat{R}/\hat{P} \leq$ K.dim $\hat{R}/M$. Combined with $\hat{P} \subseteq M$, this implies that $\hat{P} = M$. Finally $P = \hat{P} \cap R = M \cap R \supseteq \hat{Q} \cap R = Q$ implies that $P = Q$. The same reasoning is applicable if $Q \rightsquigarrow P$. ∎

Remark: The previous theorem can be generalized to the semi-prime case as follows. P must be replaced by a "homogeneous" semi-prime ideal $\bigcap_{i=1}^s P_i$; namely K.dim $R/P_i =$ K.dim R/P_1 for $i = 2, \ldots, s$. Now $N = \bigcap_{i=1}^t Q_i$ is a semi-prime ideal in R, so condition (2) must be replaced by: for every i, $1 \leq i \leq s$ there exists j, $1 \leq j \leq t$, so that $Q_j \supseteq P_i$. The proof is exactly the same, taking $P = P_1$, and showing that $P \rightsquigarrow Q$ implies $Q = P_k$. The details are omitted.

THEOREM 2.5: *Let R be a Noetherian ring satisfying the strong s.l.c. and I an ideal in R. Suppose*

(1) *R/I is a Jacobson ring,*

(2) *$I \subseteq N$, where N is a semi-prime localizable ideal in R,*

(3) *$\hat{R} \equiv \varprojlim_i R/I^i$ is Noetherian.*

Then I_N satisfies the AR property in R_N.

Proof: Given $V \rightsquigarrow W$ (or $W \rightsquigarrow V$) prime ideals in R such that $V \supseteq I$ and $\mathcal{C}(N) \subseteq \mathcal{C}(V)$, by [Br1, 3.1] we need to show that $W \supseteq I$. The proof is exactly the same as the one for Theorem 2.4. The semi-primitivity of $\hat{R}/\hat{V} \cong R/V$ (since $V \supseteq I$) is given by the Jacobson property of R/I. W is closed in the $\{I^i\}$ topology follows as in Proposition 2.2, since $I \subseteq N$. We then substitute V for P and W for Q in every step of the proof of Theorem 2.4. We conclude with $M \supseteq \mathrm{Jac}(\hat{R}) \supseteq \hat{I}$ and hence $I = \hat{I} \cap R \subseteq M \cap R = W$. ∎

We now apply Theorem 2.4 to obtain a necessary and sufficient condition for the localization of a non-maximal primitive ideal in the enveloping algebra of a complex solvable Lie algebra.

THEOREM 2.6: *Let g be a complex, solvable, finite-dimensional Lie algebra and $R \equiv Ug$ its enveloping algebra. Let P be a non-maximal primitive ideal in R. Then the following are equivalent:*

(1) *P is localizable.*

(2) *$\varprojlim_i R/P^i$ is Noetherian and P is strictly contained in a localizable prime ideal of R.*

Proof: The implication (2) $\Rightarrow$ (1) follows immediately from Theorem 2.4. We now establish the converse. Let P be a localizable prime ideal of R. By [Mc-R, p. 499] P is generated by a normalizing sequence. Now P being localizable implies by [Br2, p. 249, 8.1] that P satisfies the AR property. Consequently by [H, Remark 6.5, p. 340 and Theorem 6.5] we have that $\hat{R} \equiv \varprojlim_i R/P^i$ is noetherian. Let N be the intersection of all prime ideals which contain P properly. By [Mc-R, p. 505] and the primitivity of P, we have that $N \supset P$. So $N = P_i \cap \cdots \cap P_r$ where $\{P_i | \, i = 1, \ldots, r\}$ are the primes of R minimal with respect to strictly containing P. Let $\Gamma(g)$ be the subgroup of g^*, generated by the Jordan–Hölder values (of the adjoint representation) of g (e.g. [BdC] or [Br2]). It follows from [Br2, 2.9 (iii)], and the localizability of P, that $\tau_\lambda(P) = P$ for each $\lambda \in \Gamma(g)$, where τ_λ is the winding automorphism associated to λ (e. g. [Br2]). Consequently $\tau_\lambda(N) = N$, for each λ, which implies [Br2, 2.5] that $\{P_1, \ldots, P_r\}$ is a finite union of cliques.

Say $\{P_1, \ldots, P_k\}$, $k \leq r$, is such a clique, then $V = P_1 \cap \cdots \cap P_k$ is localizable and clearly $V \supset P$. Now [Br2, Theorem 2.13] implies that $k = 1$. ∎

3. Examples of McConnell, Jordan, Ramras and the trace ring

In the first part of this section we show that one cannot drop any of the conditions of Theorem 2.4, by using earlier examples by J. McConnell and D. Jordan, designed for related but different purposes. Surprisingly, an old example of Ramras will do as well.

In the second part of the section, we shall show how to omit condition (2) of Theorem 2.4 in case $R = T(R)$.

Example 3.1: [Mc1]. Let $A = K[v, y]$ be the polynomial ring in two variables over a field K with char $K = p > 0$. Set $R = A[x; \alpha]$, the skew polynomial ring obtained by using the K-automorphism α, given by $\alpha(v) = v + y$ and $\alpha(y) = y$. Let $P = vR + xR$. Then, as in ([Mc1] Sec. 3), $\bigcap_n P^{(n)} \neq \{0\}, \hat{R} \equiv \varprojlim_i R/P^n$ is not Noetherian and P is not localizable. However, $R/P \cong K[y]$ being semiprimitive establishes condition (1) of Theorem 2.4. Also $N \equiv yR + vR + xR$ is a maximal ideal which is generated by the centralizing sequence $\{y, v, x\}$ (the order here is important). Therefore, by standard results (e. g. [Mc2, Lemma B], [Mc3, p. 309]), N satisfies the AR property and hence is localizable. Consequently, condition (2) of Theorem 2.4 is satisfied. However, as stated above, condition (3) is not valid and P is not localizable. Also, since $Z(R) = A^\alpha[x^p]$, we have that $R = T(R)$, the trace ring of R.

Remarks: (1) McConnell described this example over any field K. The restriction char $K = p > 0$ is made in order to turn R into a P.I. ring (and actually a finite free module over its normal center).

(2) McConnell's example was originally given in order to provide an example of a Noetherian ring R with a prime ideal P so that $\hat{R}$ is not Noetherian as well as $\bigcap_n P^{(n)} \neq \{0\}$, but $\bigcap_n P^n = \{0\}$.

Example 3.2: [J1, Example 4]. Let $A = K[[v, y]]$ where K, v, y are as in the previous example and $R = A[x; \alpha]$ where α is the K-automorphism given by $\alpha(v) = v + y, \alpha(y) = y$. Let $P = vR + xR$ and $N = yR + vR + xR$. As in the previous example one sees that N is a maximal ideal with the centralizing set of generators $\{y, v, x\}$ and hence localizable. It is shown in [J1] that $\bigcap_n P^n =$

$\{0\}$, $\bigcap_n P^{(n)} \neq \{0\}$ and $\hat{R} \equiv \varprojlim_n R/P^n$ is Noetherian. Now, $\bigcap_n P^{(n)} \neq \{0\}$ implies that P is not localizable. Observe that conditions (2) and (3) of Theorem 2.4 are satisfied but (1) is not, since $R/P \cong K[[y]]$ is not semi-primitive.

Remark: Originally, D. Jordan's example was given in order to refute the implication: "$\hat{R}$ is Noetherian $\Rightarrow \bigcap_n P^{(n)} = \{0\}$", as conjectured by Small–Stafford [S²].

Surprisingly, an equivalent example could be obtained by using a variation of an example of Ramras as follows.

Example 3.3: [Ra]. Let A be a commutative regular local complete ring with 2 being a unit in A, e.g., $A = K[[x, y, z]]$. Define the quaternion algebra $R = A[1, \alpha, \beta, \alpha\beta]$ by setting $\alpha^2 = x + z^2$, $\beta^2 = y$, $\alpha\beta = -\beta\alpha$ (i. e. R is A-free with generators $1, \alpha, \beta, \beta\alpha$). It is shown in [Ra, Example 1(b), p. 351] that R is local, and gl.dim $R = 3$. Let $m \equiv (x, y, z)$. So R, being a finite(free) module over the complete commutative ring A, is complete with respect to $\{N^i\}$, where $N = zR + \alpha R + \beta R$ is its unique maximal ideal. Now if $P = (\alpha - z)R + xR + \beta R$ and $Q = (\alpha + z)R + xR + \beta R$, then P and Q are distinct prime ideals in R with $P \cap A = Q \cap A = (x, y)$. So, since $Z(R) = A$, we have that P is not localizable in R (this can be deduced from Müller's result [GW, Theorem 11.20] or directly). However $\varprojlim_n R/P^n$ is Noetherian since $\varprojlim_n R/P^n \cong R$ (the completion of a complete ring R with respect to a finer topology is equal to R). Observe that $Z(R) = A$ is a normal domain, which implies that $T(R) = R$. Hence, by [BS] or [B2], $\bigcap_n P^{(n)} \neq \{0\}$. ∎

Remark: Taking $A = K[x, y, z]$ in the previous example provides an example which could replace the one given by McConnell. We omit the rather easy details, but remark that in order to show that $\hat{R}$ is not Noetherian one can use Theorem 2.4.

Example 3.4: We now produce an example of a non-localizable prime ideal P in R such that $\bigcap_i P^i = \{0\}$, and P satisfies conditions (1), (3) of Theorem 2.4, but does not satisfy condition (2). Indeed let P be any maximal ideal in a Noetherian P.I. ring such that P is not localizable and $\bigcap_i P^i = \{0\}$ (e. g. [S²]). Clearly (1) is valid and (3) holds by [B1, Theorem 14].

Our next result, relevant to our considerations, is, surprisingly, new.

PROPOSITION 3.5: *Let R be a ring such that every semi-prime homomorphic image of R is Goldie. Let N be a localizable semi-prime ideal in $R[t]$. Then $N \cap R$ is localizable in R.*

Proof: Let $s \in \mathcal{C}(N \cap R)$ and $r \in R$. Clearly $s \in \mathcal{C}(N)$, hence $s^{-1}r = p(t)q(t)^{-1}$ where $q(t) \in \mathcal{C}(N)$. Suppose that there exists a polynomial $u(t) \in R[t]$ where the highest non-zero coefficient a of $q(t)u(t)$ is regular $\mathrm{mod}(N \cap R)$. Then $rq(t)u(t) = sp(t)u(t)$. Let b be the highest non-zero coefficient of $p(t)u(t)$. Then $ra = sb$ which implies that $s^{-1}r = ba^{-1}$, $a \in \mathcal{C}(N \cap R)$, $b \in R$. To show that such $u(t)$ exists we pass to $\bar{R} \equiv R/N \cap R \subseteq \bar{R}[t] = R[t]/(N \cap R)[t]$. Now, that $\overline{q(t)}$ is regular in $\bar{R}[t]$ follows from $\mathcal{C}(N) \subseteq \mathcal{C}((N \cap R)[t])$. Hence $\overline{q(t)}\bar{R}[t]$ is essential in $\bar{R}[t]$, so for every non-zero right ideal ρ in $\bar{R}$, we have $\overline{q(t)}\bar{R}[t] \cap \rho[t] \neq 0$. Let K be the set of all highest coefficients of elements in $\overline{q(t)}\bar{R}[t]$. Then $K \cap \rho \neq 0$ for each ρ. Now, K being a right ideal implies that K is essential in $\bar{R}$ and hence contains a regular element. ∎

The following is an interesting corollary, the converse of which seems to be open.

PROPOSITION 3.6: *Let R be a Noetherian ring and t a commuting variable. If $R[t]$ has a classical quotient ring then so does R.*

Proof: Clearly $\mathcal{C}_R(0) = \mathcal{C}_{R[t]}(0) \cap R$. Now by [S¹] we have that $\mathcal{C}_{R[t]}(0) = \mathcal{C}(V_1) \cap \cdots \cap \mathcal{C}(V_r)$ where $\{V_1, \ldots, V_r\}$ is the set of regular primes in $R[t]$. Clearly $\mathcal{C}_R(V_i \cap R) = \mathcal{C}_{R[t]}(V_i) \cap R$. Hence $\mathcal{C}_R(0) = \mathcal{C}_R(V_1 \cap R) \cap \cdots \cap \mathcal{C}_R(V_r \cap R)$. Now by assumption $V_1 \cap \cdots \cap V_r$ is localizable and by Proposition 3.5 this implies that $(V_1 \cap R) \cap \cdots \cap (V_r \cap R)$ is localizable, that is $\mathcal{C}_R(0)$ is an Ore set. ∎

We shall now show that condition (2) of Theorem 2.4 is not needed in case $R = T(R)$, the trace ring of R.

THEOREM 3.7: *Let $R = T(R)$ be a prime Noetherian P.I. ring and P a prime ideal in R with $\bigcap_n P^n = \{0\}$. Then the following statements are equivalent:*

(1) *P is localizable,*
(2) $\bigcap_n P^{(n)} = \{0\}$,
(3) *P satisfies the AR property,*
(4) $\varprojlim_i R[t]/P^i[t]$ *is Noetherian.*

Proof: Clearly (1) implies (2). That (2) implies (1) follows from [BS] or [B2, p. 86]. We next establish the implication: (4) $\Rightarrow$ (1). By Lemma 4.2, Proposition

3.5 and $T(R[t]) = T(R)[t]$, we may change notation and assume that P is semi-primitive and $\hat{R}$ is Noetherian. Clearly $R \subset \hat{R}$. So there exists a minimal prime ideal Q in $\hat{R}$, with $Q \cap R = \{0\}$. Now $\hat{P} \subseteq \mathrm{Jac}(\hat{R})$ (by Lemma 2.3) implies that $\hat{P} + Q/Q \subseteq \mathrm{Jac}(\hat{R}/Q)$. It is standard that every prime minimal over $\mathrm{Jac}(\hat{R}/Q)$ is the intersection of the primitive ideals containing it and therefore it is semi-primitive. Consequently every prime which is linked to it is semi-primitive (by [GW, Theorem 7.16]), hence it must contain $\mathrm{Jac}(\hat{R}/Q)$.

Therefore if $\{V_i/Q, i = 1, \ldots, s\}$, are the minimal primes over $\mathrm{Jac}(\hat{R}/Q)$ with the highest Krull co-dimension, then $N_1 \equiv \bigcap_{i=1}^{s}(V_i/Q)$ is localizable semi-prime ideal in $\hat{R}/Q$. Consequently $\bigcap_i (N_1)^{(i)} = \{0\}$. However, since $N_1 \supseteq \hat{P} + Q/Q$, then $\bigcap_{i=1}^{s} V_i \supseteq \hat{P}$. Also by $\hat{R}/\hat{P} \cong R/P$, we have that $V_i \cap R$ is a prime ideal in R for $i = 1, \ldots, s$ and $\mathcal{C}(N) \subseteq \mathcal{C}(N_1)$ where $N = \bigcap_{i=1}^{s}(V_i \cap R)$. Now $\mathcal{C}(N) \subseteq \mathcal{C}(N_1)$ implies that $\bigcap_i N^{(i)} = \{0\}$. Therefore by an extended version of [B2, p. 88], $N \subseteq N_2$, where N_2 is a semiprime localizable ideal, obtained by intersecting some of the primes which are minimal over N. All the conditions of Theorem 2.4 are now satisfied and P is therefore localizable.

We now prove the implication (1) $\Rightarrow$ (3). Let $p = P \cap Z(R)$. Recall that by [BW, Proposition 3] P is the unique prime in R contracting to p. Consequently, by the Going-Down property of $T(R)$ [B2, p. 82], P is the only prime of R which is minimal over pR. Hence $P^e \subseteq pR$ for some e. But pR has the AR property because it is centrally generated ([Mc-R] 4.2.6); hence P satisfies the AR property.

Finally, in order to show that (3) implies (4) we use the previous implication to conclude that $P^e[t] \subseteq p[t]R[t]$. Consequently $\varprojlim_i R[t]/P^i[t] = \varprojlim_i R[t]/p^i R[t]$ and the latter is Noetherian by standard commutative algebra arguments. ∎

4. Completion and the AR property

In this section, we first show that the (weak) AR property of I implies that $\hat{R}$ is Noetherian. We then use Theorem 2.5 to establish Theorem 4.6.

The next theorem should have been known. However, in the only reference we could find [Mc3], the same conclusion is proved with the additional assumption that the ideal has a normalizing set of generators plus other conditions (but no P.I. assumptions are required).

Recall that an ideal I satisfies the left (weak or topological) AR property, if for every left ideal L in R there exists an integer n such that $I^n \cap L \subseteq IL$.

THEOREM 4.1: *Let R be a left Noetherian P.I. ring and I an ideal in R which satisfies the left AR property. Then $\hat{R} \equiv \varprojlim_i R/I^i$ is a left Noetherian P.I. ring.*

Proof: We may assume that $\bigcap_i I^i = \{0\}$ and therefore $R \subseteq \hat{R}$. The proof follows the same pattern as [B1, Theorem 14], with a few additional changes.

Recall that, by the AR property, the topologies $\{I^i L\}, \{I^i \cap L\}$ are equivalent for every left ideal L in R. Consequently $\hat{R}L = \varprojlim_i L/I^i L = \varprojlim_i L/I^i \cap L = \text{closure}_{\hat{R}}(L) = \bigcap_i (\hat{I}^i + L)$, where the first equality is given by [Ro, p. 118] (or Lemma 2.3(2)) and the rest are as in [B1, Lemma 4]. Furthermore, by [B1, Lemma 6] $W\hat{R} \subseteq \text{closure}_{\hat{R}}(W)$, for every two-sided ideal W in R. Hence $\hat{R}W = \text{closure}_{\hat{R}}(W) \supseteq W\hat{R}$ shows that $\hat{R}W$ is a two-sided ideal in $\hat{R}$.

The rest of the argument is identical to the one in [B1, Theorem 14] and therefore the details are omitted. ∎

The next lemma is a slight generalization of a (unpublished) result due to Goodearl–Stafford).

LEMMA 4.2 (Goodearl–Stafford): *Let R be a Noetherian ring and suppose that $P \rightsquigarrow Q$ are linked prime ideals in $R[t]$. Then either $P \cap R = Q \cap R$ or $P \cap R \rightsquigarrow Q \cap R$.*

Proof: Suppose that the link $P \rightsquigarrow Q$ is given by the bimodule $B = P \cap Q/A$. Let $P_1 \equiv P \cap R$, $Q_1 \equiv Q \cap R$ and $B_1 = P_1 \cap Q_1 / A \cap R$. Note that B_1 is torsionfree on each side as $R/P_1 - R/Q_1$ bimodule. We shall consider two separate cases.

CASE 1: $B_1 \neq 0$. Then $P_1 \rightsquigarrow Q_1$ is given by the torsionfree bimodule B_1.

CASE 2: $B_1 = 0$. Hence $A \cap R = P_1 \cap Q_1$. Consequently, by considering $R/P_1 \cap Q_1 \subseteq R[t]/A$, we may assume that $P_1 \cap Q_1 = \{0\}$. Say P_1 is a minimal prime in R, then $\mathcal{C}_R(P_1)$ is an Ore set in the semiprime ring R and hence an Ore set in $R[t]$. Now it is easy to see that $\mathcal{C}_R(P_1) \subseteq \mathcal{C}_{R[t]}(P)$ and being an Ore set, we have, since $P \rightsquigarrow Q$, that $\mathcal{C}_R(P_1) \subseteq \mathcal{C}_{R[t]}(Q)$ (e.g., by [GW, Lemma 12.17]). Now $\mathcal{C}_R(Q_1) = \mathcal{C}_{R[t]}(Q) \cap R$ implies that $\mathcal{C}_R(P_1) \subseteq \mathcal{C}_R(Q_1)$ and therefore $P_1 \supseteq Q_1$ as desired. The same argument works if Q_1 is minimal. ∎

COROLLARY 4.3: *Let R be a Noetherian P.I. ring and I an ideal in R which satisfies the left AR property. Then $I[t]$ satisfies the left AR property in $R[t]$.*

Proof: Let $V \rightsquigarrow W$ be prime ideals in $R[t]$ with $V \supseteq I[t]$. By [Br1, 3.1] we need to show that $W \supseteq I[t]$. We have, by Lemma 4.2, that either $V \cap R \rightsquigarrow W \cap R$, or $V \cap R = W \cap R$. In the former case, we have, since $I \subseteq V \cap R$ and the AR property of I, that $I \subseteq W \cap R$. Hence $I[t] \subseteq W$ as needed. If $V \cap R = W \cap R$ then $I \subseteq V \cap R = W \cap R$ and therefore $I[t] \subseteq W$. ∎

The next lemma is well known.

LEMMA 4.4: *Let R be a ring, and I an ideal in R and S an Ore set in R with $I \cap S = \emptyset$. Suppose I satisfies the left AR property in R. Then I_S satisfies the left AR property in R_S.*

Recall that $S \equiv \{f \in R[t] |\, f \text{ is a non-constant monic}\}$ is a multiplicatively closed set in $R[t]$ and, by [RSS, Proposition 2.2], S is an Ore set in $R[t]$. Observe that $S \cap R = \emptyset$. We denote $R\langle t\rangle \equiv R[t]_S$. As a corollary of the previous two results we have:

COROLLARY 4.5: *Let R be a Noetherian P.I. ring and I an ideal in R which satisfies the left AR property in R. Then $I\langle t\rangle \equiv I[t]_S$ satisfies the left AR property in $R\langle t\rangle$.*

We are now able to prove the following.

THEOREM 4.6: *Let R be a local Noetherian P.I. ring and I an ideal in R. Then the following are equivalent:*

(1) *I satisfies the AR property.*

(2) *$\varprojlim_i R\langle t\rangle / I^i\langle t\rangle$ is Noetherian.*

Proof: We first assume (1). So, by Coroolary 4.5, $I\langle t\rangle$ satisfies the AR property and consequently, by Theorem 4.1, $\varprojlim_i R\langle t\rangle / I^i\langle t\rangle$ is Noetherian. Conversely assume (2) and let $N = \mathrm{Jac}(R)$. Then by Lemma 4.2, $N\langle t\rangle$ is a localizable prime ideal in $R\langle t\rangle$ with $I\langle t\rangle \subseteq N\langle t\rangle$. Also by [RSS, Theorem 2.8] $R\langle t\rangle$ is Jacobson ring. All the conditions of Theorem 2.5 are now satisfied, implying that $I\langle t\rangle_{N\langle t\rangle}$ satisfies the AR property. Let $V \supseteq I$ be a prime ideal in R with $V \rightsquigarrow W$ (or $W \hookrightarrow V$). We need to show that $W \supseteq I$. Now $V[t] \rightsquigarrow W[t]$ and $S \cap V[t] = \emptyset$ implies that $V\langle t\rangle = V[t]_S \rightsquigarrow W[t]_S = W\langle t\rangle$. Also $V\langle t\rangle \subseteq N\langle t\rangle$ as well as $W\langle t\rangle \subseteq N\langle t\rangle$ implies that $V\langle t\rangle_{N\langle t\rangle} \rightsquigarrow W\langle t\rangle_{N\langle t\rangle}$. Hence, by the AR property of $I\langle t\rangle_{N\langle t\rangle}$, we conclude that $I\langle t\rangle_{N\langle t\rangle} \subseteq W\langle t\rangle_{N\langle t\rangle}$. Consequently $I \subseteq W$. ∎

Remark: The version of Theorem 4.6 in the semi-local case requires the following addition to condition (2): Given a maximal ideal M in R with $I \subseteq M$ then $I \subseteq Q$, for every $Q \in \mathrm{clique}(M)$. We omit the proof which resembles the previous one.

5. K.dim $R/P = 1$

The main purpose of the present section is to prove the following result. We then prove Theorem 5.7.

THEOREM 5.1: *Let R be a left Noetherian P.I. ring and P a (semi) prime ideal satisfying*

(1) *P is left localizable,*

(2) *P is a finitely generated right R-module,*

(3) K.dim $R/P = 1$.

Then $\hat{R} = \varprojlim_i R/P^i$ is left Noetherian.

We will need the following:

LEMMA 5.2: *Let S be a prime left and right Goldie ring and M a finitely generated torsionfree right S-module. Let I be an ideal in S satisfying $\bigcap_i I^i = \{0\}$. Then $\bigcap_i MI^i = \{0\}$.*

Proof: Clearly $M \subseteq M \otimes_S Q(S)$, the latter being a finitely generated projective right $Q(S)$-module and therefore is embedded in a free right $Q(S)$-module G. Consequently if $g_1, \ldots, g_n$ are the free generators of G over $Q(S)$, then we can multiply $g_1, \ldots, g_n$ by $s^{-1}, s \in \mathcal{C}(0)$ such that the generators of M are S-linear combinations of $g_1 s^{-1}, \ldots, g_n s^{-1}$. An easy examination shows that $f_1 \equiv g_1 s^{-1}, \ldots, f_n \equiv g_n s^{-1}$ is a free set of elements over S. Hence $F = f_1 S + \cdots + f_n S$ is a right free S-module and clearly $M \subseteq F$. Finally $MI^i \subseteq FI^i$ for each i and $\bigcap_i FI^i = \{0\}$ since it holds at each component. ∎

COROLLARY 5.3: *Let V, W be prime ideals in a P.I. ring R and let A be an ideal with $VW \subseteq A \subset V \cap W$ such that $V \cap W/A$ is a finitely generated torsionfree right R/W-module. Suppose $0 \neq I$ is an ideal in R/W with $\bigcap_i I^i = \{0\}$. Then there exists a descending chain of ideals $\{V_i\}$ in R, $A \subset V_i \subseteq V \cap W$ for each i, and $\bigcap_i V_i = A$.*

Proof: Take $M = V \cap W/A$ and $S = R/W$, use the previous lemma and set $V_i \equiv$ the preimage of MI^i in R. ∎

Proof of Theorem 5.1: Since $\hat{R} \equiv \varprojlim_i R/P^i$ we may assume that $\bigcap_i P^i = \{0\}$, that is $R \subseteq \hat{R}$. For the sake of convenience we also assume that P is prime. Let M be a maximal member in $\mathcal{F} \equiv \{I \mid I$ is an ideal in $\hat{R}$, I is not a finitely generated left $\hat{R}$-module}. Using [B1, Lemma 11], we assume by negation that $\mathcal{F} \neq \emptyset$.

As in [B1, Lemma 12], M is a prime, non-maximal, ideal in $\hat{R}$ and $\hat{R}/M$ is left Noetherian. Consequently, by Cauchon's Theorem (e.g. [Mc-R, 13.6.15]), $\hat{R}/M$ is also right Noetherian. Also $\hat{P} = \hat{R}P = P\hat{R}$ (by Lemma 2.3), hence $\hat{P}$ is finitely generated as a left and right $\hat{R}$-module.

The proof, from now on, will proceed in steps.

STEP 1: Let V be an ideal in $\hat{R}$ satisfying $\hat{P}M \subseteq V \subset \hat{P} \cap M$ and $r\text{-ann}_{\hat{R}}(\hat{P} \cap M/V) \supset M$. Then $\hat{P} \cap M/V$ is a finitely generated left and right $\hat{R}$-module and consequently $\hat{R}/V$ is left and right Noetherian.

Indeed let $I \equiv r\text{-ann}_{\hat{R}}(\hat{P} \cap M/V)$. So, by the maximal choice of M, I is a finitely generated left $\hat{R}$-module. Furthermore $(\hat{P} \cap M)I \subseteq V$, implies that I/V is a finitely generated left $\hat{R}/\hat{P} \cap M$-module. Now $\hat{R}/\hat{P} \cap M$ is a semi-prime left and right Noetherian ring, since $\hat{R}/\hat{P} \oplus \hat{R}/M$ is a left and right Noetherian ring and is a finite central extension of $\hat{R}/\hat{P} \cap M$. Consequently $\hat{P} \cap M/V \subseteq I/V$ is a finitely generated left $\hat{R}/\hat{P} \cap M$-module. Also $\hat{P} \cap M/V$ is a finitely generated right $\hat{R}$-module since $\hat{P} \cap M/V \subseteq \hat{P}/V$, the latter being a finitely generated right $\hat{R}/M$-module, and $\hat{R}/M$ is Noetherian. Consequently, since $\hat{R}/\hat{P} \cap M$ is left and right Noetherian and $\text{Nil-}(\hat{R}/V) = \hat{P} \cap M/V$ is finitely generated as a left and right ideal, we get by standard results that $\hat{R}/V$ is left and right Noetherian.

STEP 2: $r\text{-ann}_{\hat{R}}(\hat{P} \cap M/\hat{P}M) = M$. Otherwise take $V = \hat{P}M$ in step 1 and deduce that $\hat{R}/\hat{P}M$ is left Noetherian. Consequently $M/\hat{P}M$ is a finitely generated left $\hat{R}$-module. Thus by standard results (e.g., [N, Theorem 30.6]) M is a finitely generated left R-module. A contradiction.

STEP 3: The construction of A. Let A be an ideal in $\hat{R}$ satisfying

(1) $\hat{P}M \subseteq A \subset \hat{P} \cap M$,

(2) A is maximal such that $\hat{P} \cap M/A$ is a faithful right $\hat{R}/M$-module.

The existence of A is possible by step 2 and by the fact that $\hat{P} \cap M/\hat{P}M$ is a finitely generated right, Noetherian, $\hat{R}/M$-module.

STEP 4: $\hat{P} \cap M/A$ is a torsionfree right $\hat{R}/M$-module. We firstly state an easy lemma, the proof of which is left to the reader.

LEMMA: *Let T be a right module over a prime P.I. ring S and $cx = 0$ for some $c \in T$ where x is a regular element in S. Then $c\delta = 0$ for some $0 \neq \delta \in Z(S)$.*

Now, let $C \subseteq \hat{P} \cap M/A$ be a sub-bimodule which is right torsion over $\hat{R}/M$. By the right Noetherian property of $\hat{P} \cap M/A$, we have that C is a finitely generated $\hat{R}/M$-module, $C = c_1\hat{R}/M + \cdots + c_r\hat{R}/M$. Let $0 \neq \delta_i \in Z(\hat{R}/M)$ so that $c_i\delta_i = 0$, $i = 1, \ldots, r$ and set $d = \delta_i \cdots \delta_r$. Then clearly $c_i d = 0$ for $i = 1, \ldots, r$ and $Cd = 0$ (since d is central). Consequently $r\text{-ann}_{\hat{R}}(C) \supset M$. Now $C = C_1/A$, $C_1 \supset A$, so by the maximality of A (step 3) $\hat{P} \cap M/C_1$ is an unfaithful right $\hat{R}/M$ module. Hence, there exists $x \in \hat{R} - M$ satisfying $(\hat{P} \cap M)x \subseteq C_1$. Also, since $r\text{-ann}_{\hat{R}}(C) \supset M$, there exists $s \in \mathcal{C}(M) \cap r\text{-ann}_{\hat{R}}(C)$. Therefore, $(\hat{P} \cap M)xs \subseteq C_1 s \subseteq A$. Consequently $xs \in r\text{-ann}_{\hat{R}}(\hat{P} \cap M/A) = M$ (by step 2). Therefore, we get, since $s \in \mathcal{C}(M)$, that $x \in M$. An obvious contradiction.

STEP 5: A is closed in the $\{\hat{P}^i\}$ topology. That is $A = \bigcap_i (A + \hat{P}^i)$. Consider the bimodule $\hat{P} \cap M/A$. Clearly if $z \neq 0$ is a non-invertible central element in $\hat{R}/M$, then $\bigcap_i z^i \hat{R}/M = \{0\}$. So by Corollary 5.3, there are ideals $\{V_i\}$ in $\hat{R}$, satisfying $A \subset V_i \subseteq V_{i-1} \subseteq \hat{P} \cap M$ and $\bigcap_i V_i = A$. By steps 1 and 3, $\hat{R}/V_i$ is a left and right Noetherian P.I. ring for each i. Consequently by Jategaonkar's Theorem [GW, Theorem 12.8], $\bigcap_n \mathrm{Jac}^n(\hat{R}/V_i) = \{0\}$, for each i. Now $\hat{P} + V_i/V_i \subseteq \mathrm{Jac}(\hat{R}/V_i)$, implies that $\bigcap_n (\hat{P}^n + V_i) = V_i$, for each i. Hence $\bigcap_n (\hat{P}^n + A) \subseteq \bigcap_i \{\bigcap_n (P^n + V_i)\} = \bigcap_i V_i = A$.

STEP 6: Separation into cases. Let $Q \equiv M \cap R$. Recall that $\hat{R}/M$ is left and right Noetherian and $\hat{P} + M/M \subseteq \mathrm{Jac}(\hat{R}/M)$. Hence, $\bigcap_i \mathrm{Jac}^i(\hat{R}/M) = \{0\}$ and therefore $\cap_i (\hat{P}^i + M) = M$. Consequently by [B1, Theorem 13(i)] $Q \equiv M \cap R$ is a prime ideal in R. Recall that $\hat{P} \cap R = P$. We now need to deal with two separate cases, namely Case (1): $P \cap Q \supset A \cap R$, and Case (2): $P \cap Q = A \cap R$.

STEP 7: Case (1). We have $P \cap Q \supset A \cap R$. Recall that by [B1, Theorem 13(ii)] $\text{p.i.deg.}(R/Q) = \text{p.i.deg.}(\hat{R}/M)$ and consequently $\mathcal{C}(Q) \subseteq \mathcal{C}(M)$. Therefore, by Step 4, $P \cap Q/A \cap R$ is a torsionfree right R/Q-module. Now R/Q is a prime left Noetherian P.I. ring, hence, by Cauchon's Theorem, a right Noetherian ring. Consequently, since $P/A \cap R$ is a right finitely generated R/Q-module, we have that $P/A \cap R$ is a right Noetherian R-module and therefore $P \cap Q/A \cap R$, being a submodule, is right Noetherian as well. Furthermore, $R/P \cap Q$ being left Noetherian and semi-prime, is right Noetherian. Thus $R/A \cap R$ is a left and right Noetherian P.I. ring and $B = P \cap Q/A \cap R$ is a finitely generated $R/P - R/Q$ bimodule. Consequently, by [GW, Theorem 12.4] (applied to $R/A \cap R$), $\text{K.dim } R/\ell\text{-ann}_R B = \text{K.dim } R/r\text{-ann}_R B$. By step 2 and the above, we have that $Q = r\text{-ann}_R B$. Set

$K \equiv \ell\text{-ann}_R B$. Clearly $K \supseteq P$. If $K \supset P$, then K is co-Artinian and therefore K.dim $R/Q = 0$, that is Q is maximal. Now $\bigcap_i(\hat{P}^i + M) = M$ implies that $\bigcap_i(P^i+Q) = Q$. So, by the maximality of Q, we have $P \subseteq Q$. Hence $\hat{P} \subseteq \hat{Q} \subseteq M$ and therefore M is left finitely generated. If $K = P$ then $P/A \cap R \rightsquigarrow Q/A \cap R$ is a link in the two-sided Noetherian P.I. ring $R/A \cap R$. So, the left localizability of P implies that $P/A \cap R = Q/A \cap R$. Consequently $P = Q$ and, therefore $\hat{P} = \hat{Q} \subseteq M$ and we finish as before.

STEP 8: Case (2). We have $P \cap Q = A \cap R$. Consequently $QP \subseteq A$ and therefore $\widehat{QP} \subseteq \hat{A} = A$, where the last equality is due to step 5. However, by Lemma 2.3, $\hat{Q}\hat{P} \subseteq \widehat{QP}$ implies $\hat{Q}\hat{P} \subseteq A$. We need the following, where [,] denotes the commutator:

CLAIM: Let $z \in \hat{P}$ and $[z, \hat{R}] \subseteq \hat{Q}$. Then $[z^2, \hat{R}] \subseteq A$. Indeed $[z^2, \hat{R}] \subseteq z[z, \hat{R}] + [z, \hat{R}]z \subseteq \hat{P}\hat{Q} + \hat{Q}\hat{P} \subseteq A$.

We shall show now that $P \subseteq Q$. Suppose by negation that $P \nsubseteq Q$. We can find $z \in P - Q$ with $[z, R] \subseteq Q$ (that is $\bar{z}$ is central in R/Q). Consequently $\bar{z}$ is central in $\widehat{(R/Q)} = \hat{R}/\hat{Q}$. Hence, if z_1 denotes the image of z in $\hat{R}/A$, we clearly have that $z_1^2(\hat{P} \cap M/A) = 0$. Hence, by the Claim, $(\hat{P} \cap M/A)z_1^2 = 0$. Consequently $z^2 \in r\text{-ann}_{\hat{R}}(\hat{P} \cap M/A) = M$. So $z^2 \in M \cap R = Q$. This, together with the centrality of $\bar{z}$ in R/Q, imply that $\bar{z} = 0$, that is $z \in Q$. An obvious contradiction. Finally $P \subseteq Q$ implies $\hat{P} \subset \hat{Q} \subseteq M$. Hence, M is a finitely generated left $\hat{R}$-module. ∎

Remarks: 1. It is remarkable that in case (2) we neither used the left localizability of P nor the K.dim $R/P = 1$ assumption.

2. It would be nice to remove the right finite generation assumption on P.

3. Theorem 5.1 is valid for P a semi-prime ideal. The proof is practically the same.

4. The condition: "P is localizable", is not necessary for the conclusion of Theorem 5.1. Indeed, let M be any maximal ideal in a P.I. Noetherian ring R with $\bigcap_i M^i = \{0\}$, but M is not localizable. Then by [B1, Theorem 14] we have that $\hat{R}$ is Noetherian. By some modifications of the proof one can prove that actually $\varprojlim_i R[t]/M^i[t]$ is Noetherian. So $P = M[t]$ satisfies: K.dim $R[t]/P = 1$, $\varprojlim_i R[t]/P^i$ is Noetherian but P is not localizable (otherwise M would be).

PROPOSITION 5.4: *Let R be a Noetherian P.I. ring and P a prime ideal in R. Suppose:*

(1) *P is left localizable,*

(2) *$\hat{R} \equiv \varprojlim_i R/P^i$ is Noetherian.*

Then $\hat{P}$ is a left localizable ideal in $\hat{R}$.

Proof: We may assume that $\bigcap_i P^i = \{0\}$ and hence $R \subset \hat{R}$. Now (2) implies that every two-sided ideal in $\hat{R}$ is closed with respect to $\{\mathrm{Jac}^i(\hat{R})\}$, and consequently with respect to $\{\hat{P}^i\}$ since $\hat{P} \subseteq \mathrm{Jac}(\hat{R})$.

Let Q_1 be a prime ideal in $\hat{R}$ with $\hat{P} \rightsquigarrow Q_1$. We shall show that $\hat{P} = Q_1$. Clearly by the previous paragraph, Q_1 is closed. Consequently by [B1, Proposition 13], $Q \equiv Q_1 \cap R$ is a prime ideal in R and $\text{p.i.deg}(R/Q) = \text{p.i.deg}(\hat{R}/Q_1)$. Hence $\mathcal{C}_R(Q) \subseteq \mathcal{C}_{\hat{R}}(Q_1)$. Let A be a two-sided ideal in $\hat{R}$ satisfying $\hat{P}Q_1 \subseteq A \subset \hat{P} \cap Q_1$ and $\hat{P} \cap Q_1/A$ is $\hat{R}/\hat{P} - \hat{R}/Q_1$ torsionfree.

Suppose firstly that $R \cap A \subset P \cap Q$. Then, since $R/P \cong \hat{R}/\hat{P}$ and $\mathcal{C}_R(Q) \subseteq \mathcal{C}_{\hat{R}}(Q_1)$, we have that $P \cap Q/R \cap A$ is $R/P - R/Q$ torsionfree, which implies by the left localizability of P that $P = Q$. Hence $\hat{P} = \hat{Q} \subseteq Q_1$. Now the equality $Q_1 = \hat{P}$ follows from $\text{K.dim}\ \hat{R}/\hat{P} = \text{K.dim}\ \hat{R}/Q_1$.

Next, suppose that $R \cap A = P \cap Q$. Then $QP \subseteq A$ and hence, since A is closed in $\hat{R}$, $\widehat{QP} \subseteq A$. The rest of the argument is identical to the proof of Theorem 5.1, step 8. Consequently $P \subseteq Q$ and therefore $\hat{P} \subseteq \hat{Q} \subseteq Q_1$. We conclude as in the previous case. ∎

Remark: A similar proof, in which P is assumed to satisfy the left AR property, implies that $\hat{P}$ satisfies the left AR property.

We next have the following corollary.

LEMMA 5.5: *Let R be a Noetherian P.I. ring and P a prime ideal satisfying*

(1) *P is localizable*

(2) $\text{K.dim}\ R/P = 1$.

Then $\hat{P}$ satisfies the AR property in $\hat{R}$.

Proof: $\hat{R}$ is Noetherian by Theorem 5.1. Let $V \supseteq \hat{P}, V \rightsquigarrow W$ (or $W \rightsquigarrow V$) where V, W are prime ideals in R. If $V = \hat{P}$ then the previous proposition shows that $W = \hat{P}$. If $V \supset \hat{P}$ then V is maximal, that is $\text{K.dim}\ \hat{R}/V = 0$. Consequently, by [GW, Theorem 13.15] we have that $\text{K.dim}\ \hat{R}/W = 0$, that is W is maximal. Now $\hat{P} \subseteq \mathrm{Jac}(\hat{R})$ forces $\hat{P} \subseteq W$. ∎

LEMMA 5.6: *For every ideal I we have*

$$\lim_{\leftarrow i} R[t]/I^i[t] \cong \lim_{\leftarrow i} \hat{R}[t]/\hat{I}^i[t].$$

Proof: This is a consequence of the isomorphisms $R/I^i[t] \cong R[t]/I^i[t]$ and $\hat{R}[t]/\hat{I}^i[t] \cong \hat{R}/\hat{I}^i[t]$. ∎

We now have the following characterization.

THEOREM 5.7: *Let R be a Noetherian P. I. ring and P a prime ideal with K.dim $R/P = 1$. Then the following are equivalent:*

1. *P is localizable,*
2. (i) *$P[t] \subseteq N$, N is a localizable semi-prime in $R[t]$, and*
   (ii) *$\lim_i R[t]/P^i[t]$ is Noetherian.*

Proof: That (2) implies (1) follows from Theorem 2.4. Given (1), then by Theorem 5.1 we have that $\hat{R}$ is Noetherian. Consequently, $\hat{P}$ satisfies the AR property by Lemma 5.5. Hence, by Corollary 4.3, $\hat{P}[t]$ satisfies the AR property in $\hat{R}[t]$. Therefore $\lim_i \hat{R}[t]/\hat{P}^i[t]$ is Noetherian (by Theorem 4.1). Now by Lemma 5.6, (ii) is established. ∎

6. Localization does not always imply noetherian completion

Let R be a finitely generated (over a central subring) Noetherian prime P. I. ring with $T(R) \neq R$. We have $T(R) = RZ(T(R)) = R[\alpha_1, \dots, \alpha_n]$ where $Z(T(R)) = Z(R)[\alpha_1, \dots, \alpha_n]$. There exists a natural onto map

$$\nu\colon R[t_1, \dots, t_n] \to R[\alpha_1, \dots, \alpha_n] = T(R), \text{ via}$$
$$t_i \to \alpha_i.$$

Also denote by ν_1 the restriction of ν to $Z(R)[t_1, \dots, t_n]$. Clearly ν_1 is an onto map

$$\nu_1\colon Z(R)[t_1, \dots, t_n] \to Z(r)[\alpha_1, \dots, \alpha_n] = Z(T(R)).$$

We denote by $I \equiv \ker \nu$, $a = \ker \nu_1$ and $R[t_1, \dots, t_n] \equiv R[\underline{t}]$. for every prime ideal p in $Z(T(R))$ we clearly have Nil-$(pT(R)) = \bigcap_{i=1}^S P_i$, where $P_1, \dots, P_S$ are the prime ideals in $T(R)$ minimal over $pT(R)$. It is corollary of the Going-Down between $Z(T(R))$ and $T(R)$ [B2, p. 82] that $P_i \cap Z(T(R)) = p$, for $i = 1, \dots, s$. We denote by $m \equiv \nu_1^{-1}(p)$ and $M_i \equiv \nu^{-1}(P_i)$, for $i = 1, \dots, s$.

The next result is crucial.

PROPOSITION 6.1: *Let R, $R[\underline{t}]$, ν, ν_1, p, m, M_i, I, be as above. Then*

(1) *I is a prime ideal in $R[\underline{t}]$ with* $\text{p.i.deg}(R[\underline{t}]/I) = \text{p.i.deg}(R)$;

(2) *if I satisfies the AR property, then $N \equiv \bigcap_{i=1}^{s} M_i$, is localizable for every prime p.*

Proof: (1) is clear. To prove (2), suppose that I satisfies the AR property. Fix i and suppose that V is a prime ideal with $M_i \rightsquigarrow V$. Now $I = \ker \nu \subseteq M_i$, implies, by [GW, Proposition 11.16] and the AR property of I, that $I \subseteq V$. Also $M_i \rightsquigarrow V$ implies by [GW, Lemma 11.7], that $M_i \cap Z(R[\underline{t}]) = V \cap Z(R[\underline{t}])$. Recall that $Z(R[\underline{t}])/a \cong Z(T(R))$. We next observe that $M_i \cap Z(R[\underline{t}]) = m$. Indeed let $x \in M_i \cap Z(R[\underline{t}])$, then $y \equiv \nu(x) \in \nu(M_i) \cap \nu(Z(R[\underline{t}]) = P_i \cap Z(T(R)) = p$. So since $x \in Z(R[\underline{t}]) = Z(R)[\underline{t}]$, we have that $x \in \nu_1^{-1}(p) = m$. The reverse inclusion, that $m \subseteq M_i \cap Z(R[\underline{t}])$, is trivial. Thus $V \cap Z(R[\underline{t}]) = m$. Hence $\overline{V} \equiv V/I$, satisfies $\overline{V} \cap Z(T(R)) = p$ (using $m/a = p$). consequently since $\overline{V}$ is a prime ideal in $T(R)$, we have that $\overline{V} = P_j$, for some $j, 1 \leq j \leq s$. Hence $V = M_j$ for some j. ∎

We now have an important Corollary.

COROLLARY 6.2: *Let R be a Noetherian prime finitely generated P. I. ring and p a prime ideal in $Z(T(R))$. Let $\text{Nil-}(pR) = \bigcap_{i=1}^{s} P_i$ and suppose that $K \equiv \bigcap_{i=1}^{s}(P_i \cap R)$ is not localizable. Then $I \equiv \ker \nu$ does not satisfy the AR property.*

Proof: Suppose by negation that I satisfies the AR property. Then by Proposition 6.1, and continuing with the notation used there, $N = \bigcap_{i=1}^{s} M_i$ is localizable in $R[\underline{t}]$. Consequently, by Proposition 3.5, $N \cap R = \bigcap_{i=1}^{s}(M_i \cap R)$ is localizable in R. Now it is easy to check that $M_i \cap R = (M_i/I) \cap R = P_i \cap R$, for each i and consequently $K = N \cap R$ is localizable, a contradiction. ∎

Remark 6.3: It is very easy now to produce a prime Noetherian P. I. ring S, finite module over its Noetherian center with I, a prime ideal in S, $\text{p.i.deg}(S) = \text{p.i.deg}\, S/I$, but I does not satisfy the AR property. Indeed take any prime Noetherian P. I. right R, finite over its center, with P a maximal ideal such that $\bigcap_i P^i = \{0\}$, but P is not localizable. Then by [B2, Theorem 9] there exists a maximal ideal p in $Z(T(R))$ such that $P_i \cap R = P$, for $i = 1, \ldots, s$ where $\bigcap_{i=1}^{s} P_i = \text{Nil-}(pT(R))$. Then $S = R[\underline{t}]$, and $I = \ker \nu$ will do. To find such an R we either take R from [S^2] or as in [BS].

We can make S semilocal by localizting it at a suitable maximal clique with (at least) one of its memebers containing I. However, we would like to have a local example with some special features as in Example 6.6.

In order to achieve this we need the following.

PROPOSITION 6.4: *Let $R \subseteq S$ be Noetherian P. I. rings and K and ideal in S with $K \subseteq R$. Let $V_1 \neq V_2$, be prime ideals in S where $K \not\subseteq V_i$, for $i = 1, 2$. Then $V_1 \rightsquigarrow V_2$ implies $V_1 \cap R \rightsquigarrow V_2 \cap R$.*

Proof: Let $V_1 \cap V_2/A$ be a torsionfree $S/V_1 - S/V_2$ bimodule given by $V_1 \rightsquigarrow V_2$. Observe that $R/V_i \cap R \cong R + V_i/V_i$ has the same quotient ring as S/V_i, since $K + V_i/V_i$ is a common ideal, for $i = 1, 2$. Hence $V_i \cap R$ is a prime ideal in R, $i = 1, 2$. If $(V_1 \cap R) \cap (V_2 \cap R) \neq A \cap R$ then $V_1 \cap R \rightsquigarrow V_2 \cap R$ as needed. Suppose by negation that $(V_1 \cap R) \cap (V_2 \cap R) = A \cap R$, hence $(V_2 \cap R)(V_1 \cap R) \subseteq A$. Also $V_2 \cap K \not\subseteq V_1 \cap R$ because $V_2 \cap K \subseteq V_1$ implies $V_2 \subset V_1$ ro $K \subset V_1$, an obvious contradiction. So, there exists $\delta \in V_2 \cap K$, with $0 \neq \overline{\delta} \in Z(R/V_1 \cap R)$. Since $R/V_1 \cap R$ and S/V_1 have the same quotient ring we have that $\overline{\delta} \in Z(S/V_1)$, that is $[\delta, S] \subseteq V_1$. Now $\delta^2[\delta, S] \subseteq \delta^2 V_1 \subseteq \delta(KV_1) \subseteq \delta(K \cap V_1) \subseteq (V_2 \cap R)(V_1 \cap R) \subseteq A$. Consequently $[\delta^3, S] \subseteq \delta^2[\delta, S] + [\delta^2, S]\delta \subseteq A + [\delta^2, S]\delta \subseteq A + V_1\delta \subseteq A + V_1(V_2 \cap K) \subseteq A + V_1V_2 \subseteq A$. Let $\bar{S} \equiv S/A$, then $\overline{\delta^3}$ is central in $\bar{S}$, moreover $\ell\text{-ann}_{\bar{S}} V_1 \cap V_2/A = V_1/A \equiv \bar{V}_1$ and $r\text{-ann}_{\bar{S}} V_1 \cap V_2/A = V_2/A = \bar{V}_2$. Now $\overline{\delta^3} \in \bar{V}_2$, so by the centrality of $\overline{\delta^3}$, we have $\overline{\delta^3} \in \bar{V}_1$ that is $\delta^3 \in V_1$. Since $\bar{\delta}$ is central in S/V_1, this implies $\delta \in V_1$, a contradiction. ∎

COROLLARY 6.5: *Let R, S, V_1, V_2, K be as in the previous proposition and P a prime ideal in S such that $P \subset V_1, P \not\subseteq V_2$. Then*

(1) *$P \cap R$ does not satisfy the AR property in R, and*
(2) *$P \cap R$ is a prime ideal in R and* $\text{p.i.deg}(R/P \cap R) = \text{p.i.deg}(S/P)$.

Proof: That $P \cap R$ does not satisfy the AR property follows from the previous result. Indeed if $P \cap R \subset V_2$, then $P \cap K \subseteq V_2$, which leads to an obvious contradiction. Now, $R + P/P$ and S/P have $K + P/P$ as a common non-trivial ideal. Hence $R + P/P \cong R/P \cap R$ is a prime ring with the same p.i. degree as S/P. ∎

The construction of Example 6.6

Let A be an affine prime P.I. ring finite module over its center with the following properties:

(1) K.dim $A = 1$,

(2) $\bigcap_i M^i = \{0\}$ for some maximal ideal M which is not localizable.

The existence of such is given by e.g. [BS, Remark 2, p. 330]. Let z be a new variable and $B = A[z]$. Now, set

$$S = B[t_1, \ldots, t_n] \quad \text{where } T(B) = B[\alpha_1, \ldots, \alpha_n]$$

and

$$\nu\colon S \to T(B), \quad \text{via}$$
$$t_i \to \alpha_i$$

is the canonical homomorphism as described at the beginning of the section. Also we denote $I \doteq \ker \nu$. So, by Remark 6.3, I does not satisfy the AR property. Now, by [B2, Theorem 9] (or by direct calculation), there exists a clique $\{P'_1, \ldots, P'_s\}$ in $T(A)$ with $P'_i \cap A = M$ for $i = 1, \ldots, s$. Consequently, using $T(B) = T(A)[z], \{P'_1[z], \ldots, P'_s[z]\}$ is a clique in $T(B)$ with $P'_i[z] \cap B = M[z]$ for each i. Set $P_i \equiv P'_i[z]$, and $M_i \equiv \nu^{-1}(P_i)$ for $i = 1, \ldots, s$. Therefore $(\bigcap_{i=1}^s P_i) \cap B = M[z]$ and $M[z]$ is not localizable, since M is not localizable. Now, if $\bigcap_{i=1}^s M_i$ is localizable in S, then, by Proposition 3.5, $(\bigcap_{i=1}^s M_i) \cap B$ is localizable, but $(\bigcap_{i=1}^s M_i) \cap B = (\bigcap_{i=1}^s P_i) \cap B = M[z]$, which leads to a contradiction. Consequently, there exists a prime ideal V in S with $M_j \rightsquigarrow V$ (or $V \rightsquigarrow M_j$) for some $1 \leq j \leq s$, but $V \neq M_i$ for each i. If $I \subset V$ then the proof of Proposition 6.1 shows that $V = M_i$ for some i. Hence $I \not\subseteq V$. We denote by $M_j \equiv V_1$ and $V \equiv V_2$. Let $N_1 \supset V_1$ be a maximal ideal and $\{N_1, \ldots, N_t\} =$ clique(N_1). Set $N = \bigcap_{i=1}^s N_i$. It is clear that $\mathcal{C}(N) \subseteq \mathcal{C}(V_1)$, hence by [GW, Lemma 12.7], $\mathcal{C}(N) \subseteq \mathcal{C}(V_2)$, which implies that $V_2 \subseteq N_j$, for some $1 \leq j \leq t$.

Finally set $R = Z(S) + N$. We need to show that R satisfies all the required properties. First, since $Z(R) = Z(S)$, we have that R is a prime affine P.I. ring which is a finite module over its Noetherian center. Next, N is semi-maximal in S (and is a maximal ideal in R), so being in addition localizable in S, it satisfies the AR property in S. This immediately implies that N satisfies the AR property in R, and so, by the maximality of N in R, we have that N is localizable in R. Now, by Corollary 6.5, the prime ideal $I \cap R$ does not satisfy the AR property in R and clearly $I \cap R \subseteq V_1 \cap R \subseteq N_1 \cap R = N$. Furthermore $(I \cap R)_N$ does not satisfy the AR property in the local ring R_N since $\mathcal{C}_S(N) \subseteq \mathcal{C}_S(V_2)$ an therefore $\mathcal{C}_R(N) \subseteq \mathcal{C}_R(V_2 \cap R)$ which shows that $(V_1 \cap R)_N \rightsquigarrow (V_2 \cap R)_N$, but $(I \cap R)_N \nsubseteq (V_2 \cap R)_N$ although $(I \cap R)_N \subseteq (V_1 \cap R)_N$. Also $\overleftarrow{\lim}_i R/(I \cap R)^i$ is not Noetherian by Theorem 2.5. The rest of the properties of Example 6.6, as stated in the introduction, can now be easily shown. ∎

To clarify the AR property notion, we add the following.

LEMMA 6.7: *Let R be a finite module over its Noetherian center and P is a prime ideal in R. Set $p = P \cap Z(R)$. Then P satisfies the AR property iff $P^k \subseteq pR$ for some k.*

Proof: Say $P^k \subseteq pR$. Given a left (right) ideal L in R we have $p^e R \cap L \subseteq pL$ for some e. Consequently $P^{ek} \cap L \subseteq p^e R \cap L \subseteq pL \subseteq PL$. Conversely, suppose P satisfies the AR property and $V \supseteq pR$ a minimal prime. If $V \neq P$, then, by Müller's result [GW, Theorem 11.20], since P is localizable, $V \cap Z(R) \equiv v \supset p$. Therefore, by the Going-Up property, there exists a prime ideal W in R, with $W \supset P$ and $W \cap Z(R) = v$. Now, $V \cap Z(R) = W \cap Z(R)$, implies that $V \in$ clique(W) (by Müller's result). Therefore by [GW, Proposition 11.16] we have that $P \subseteq V$ as needed. ∎

Remark 6.8: Example 6.6 is also a P.I. counterexample, to a question of Small–Stafford [S²]. P being localizable implies that $\bigcap_n P^{(n)} = \{0\}$. A non-P.I. example is already given in [J2]. Consequently, Example 6.6 provides a counterexample to a question of Jordan [J2, Remark 2, p. 233].

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